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Combined forced and free convection with viscous dissipation in a vertical circular duct

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Abstract

Laminar flow heat transfer in a vertical circular duct is investigated by taking into account both viscous dissipation and the effect of buoyancy. The temperature on the duct wall is considered as uniform and the velocity field is assumed to be parallel. A perturbation method is employed to solve the momentum balance equation and the energy balance equation. The radius of convergence of the perturbation series is evaluated by means of Domb–Sykes plots. A comparison with the velocity and temperature profiles in the case of laminar forced convection with viscous dissipation is performed in order to point out the effect of buoyancy. The case of convective boundary conditions is also discussed.

Nomenclature

Bi $=h_e R_0/k$, Biot number
Br Brinkman number defined by equation (26)
c_p specific heat at constant pressure
g gravitational acceleration
Gr $=8R_0^3 g\beta\Delta t/\nu^2$, Grashof number
h_e external convection coefficient
j non-negative integer number
k thermal conductivity
M mass flow rate
M_0 $=\pi k/(4g\beta)$, reference mass flow rate
n non-negative integer number
Nu Nusselt number defined by equation (24)
p pressure
P $=p+\rho_b gX$, difference between the pressure and the hydrostatic pressure
r $=R/R_0$, dimensionless radial coordinate
R radial coordinate
Re $=2R_0 U_m/\nu$, Reynolds number
R_0 radius of the tube
S_n partial sum defined by equation (57)
T temperature
T_b bulk temperature
T_e reference temperature of the external fluid
T_0 wall temperature
u $=U/U_m$, dimensionless axial velocity
U axial component of the velocity field
U_m mean axial velocity
$\mathbf{U}$ velocity field
X axial coordinate.

Greek symbols

α $=k/(\rho_b c_p)$, thermal diffusivity
β coefficient of thermal expansion
ΔT $=\mu U_m^2/k$, reference temperature difference
η $=(T_b-T_0)/\Delta T$, temperature difference ratio
η_e $=(T_b-T_e)/\Delta T$, external temperature difference ratio
θ $=(T-T_b)/\Delta T$, dimensionless temperature
λ $=-(\mathrm{d}P/\mathrm{d}X)R_0^2/(\mu U_m)$, dimensionless pressure drop
μ dynamic viscosity
ν $=\mu/\rho_b$, kinematic viscosity
Ξ $=Gr/Re$, dimensionless parameter
ρ mass density
ρ_b value of ρ for $T=T_b$.

1. Introduction

Several papers deal with the effect of frictional heating on the laminar forced convection in circular tubes. One

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of the earliest investigations on this subject can be found in the paper by Brinkman [1]. In this paper, the effect of viscous dissipation is analyzed with reference to thermally developing forced convection in a pipe either with a uniform boundary temperature or with an insulated boundary. Ou and Cheng [2] present a solution for the temperature field in the thermal entrance region of a circular duct with a uniform boundary heat flux. Lin et al. [3] obtain the temperature field and the local Nusselt number for thermally developing forced convection in a pipe with convective boundary conditions. Moreover, the temperature field and the local Nusselt number are determined for thermally developing heat transfer by Basu and Roy [4], both in the case of a uniform boundary heat flux and in the case of a uniform boundary temperature. One of the most interesting results pointed out in refs. [3, 4] is that, both for convective boundary conditions and for a uniform boundary temperature, the Nusselt number for laminar and thermally developed forced convection in a circular tube is $48/5 = 9.6$. Obviously, this result holds whenever frictional heating is taken into account, even if the Brinkman number is very small. As a consequence, the viscous dissipation effect can never be neglected in the evaluation of the Nusselt number in the thermally developed region. In fact, as is well known, in the absence of viscous dissipation, the thermally developed value of the Nusselt number is 3.6568 for uniform boundary temperature [5], while it depends on the Biot number for convective boundary conditions [6, 7].

The aim of the present paper is to extend the results for the fully developed region obtained in refs. [3, 4], by taking into account the effect of buoyancy. Indeed, for steady and laminar mixed convection in a vertical circular tube, the momentum and energy balance equations are coupled with each other and the viscous dissipation term influences not only the temperature field but also the velocity field.

In the literature, several studies on the influence of frictional heating in free convection are available as, for instance, the papers by Gebhart [8], Turcotte et al. [9] and Soundalgekar et al. [10]. Moreover, some papers deal with the internal mixed convection with viscous dissipation. For instance, Iqbal et al. [11] consider the fully-developed combined forced and free convection in a vertical circular tube with a uniform boundary heat flux. These authors employ three different mathematical techniques for the solution of the coupled momentum and energy balance equations: an extended Frobenius method, the Galerkin method and the Runge–Kutta method. Rokerya and Iqbal [12] analyze mixed convection in a vertical annular duct with a uniform heat flux either on the internal surface or on the external surface. They obtain the velocity profiles and the temperature profiles by the Runge–Kutta fourth-order method. More recently, Barletta [13] and Zanchini [14] have investigated the effect of viscous dissipation for mixed convection in a parallel-plate vertical channel. In particular, both the case of symmetric or asymmetric temperatures prescribed at the channel walls [13] and the case of convective boundary conditions [14] have been studied. In these papers, a perturbation method is employed to solve the momentum balance and energy balance equations.

To the author's knowledge, no solution of the problem of mixed convection with viscous dissipation in a vertical circular duct with uniform boundary temperature is available in the literature. In the present paper, this problem will be solved by means of a perturbation method. In particular, the velocity and temperature profiles will be expressed as perturbation series with respect to a dimensionless parameter which accounts for the buoyancy effect.

2. Formulation of the problem and governing equations

Let us consider a Newtonian fluid which steadily flows in an infinitely long vertical tube with radius R_0. The flow is assumed to be laminar and parallel, i.e. the only non-vanishing component of the velocity field $\mathbf{U}$ is the axial component U. The temperature field on the boundary $R = R_0$ is uniform with a value T_0. As it is shown in Fig. 1, the axial coordinate X is directed upward, i.e. it has a direction opposite to the gravitational acceleration vector. Since only the axial component of $\mathbf{U}$ is non-zero, the mass balance equation ensures that

$$\frac{\partial U}{\partial X} = 0. \qquad (1)$$

As a consequence of equation (1), U depends only on the radial coordinate R. Let us assume the equation of state

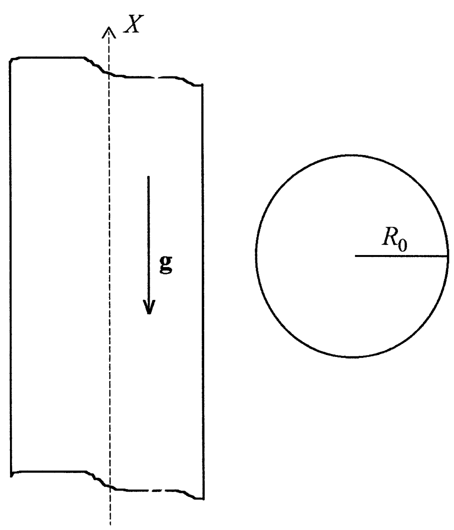

Fig. 1. Drawing of the system.

$$\rho = \rho_b[1-\beta(T-T_b)] \tag{2}$$

as well as the Boussinesq approximation. In equation (2), the reference temperature is the bulk temperature T_b defined as

$$T_b = \frac{2}{U_m R_0^2}\int_0^{R_0} UTR\,dR \tag{3}$$

where the mean velocity U_m is given by

$$U_m = \frac{2}{R_0^2}\int_0^{R_0} UR\,dR. \tag{4}$$

Then, the axial and radial components of the momentum balance equation can be written as [15]

$$\beta g(T-T_b) - \frac{1}{\rho_b}\frac{\partial P}{\partial X} + \frac{\nu}{R}\frac{d}{dR}\left(R\frac{dU}{dR}\right) = 0 \tag{5}$$

$$\frac{\partial P}{\partial R} = 0 \tag{6}$$

where $P = p + \rho_b g X$ is the difference between the pressure and the hydrostatic pressure. As a consequence of equation (6), P depends only on the axial coordinate X. Equation (5) yields

$$\frac{\partial T}{\partial X} = \frac{dT_b}{dX} + \frac{1}{\beta g \rho_b}\frac{d^2P}{dX^2}. \tag{7}$$

Equation (7) implies that $\partial T/\partial X$ does not depend on R. Since $\partial T/\partial X$ is zero at the boundary $R = R_0$, then $\partial T/\partial X$ vanishes everywhere and the temperature T depends only on R. Therefore, equation (3) implies that T_b does not depend on X and equation (7) allows one to conclude that

$$\frac{d^2P}{dX^2} = 0 \tag{8}$$

i.e. that dP/dX is a constant. The energy balance equation can be written as [15]

$$\frac{\alpha}{R}\frac{d}{dR}\left(R\frac{dT}{dR}\right) + \frac{\nu}{c_p}\left(\frac{dU}{dR}\right)^2 = 0. \tag{9}$$

The boundary conditions on the velocity U and on the temperature T are

$$U(R_0) = 0, \quad T(R_0) = T_0. \tag{10}$$

Moreover, both U and T must not be affected by singularities at $R = 0$. This condition can be expressed as

$$\left.\frac{dU}{dR}\right|_{R=0} = 0, \quad \left.\frac{dT}{dR}\right|_{R=0} = 0. \tag{11}$$

Note that, as a consequence of equations (2)–(4), the mass flow rate M can be expressed as

$$M = 2\pi\int_0^{R_0} \rho UR\,dR$$

$$= 2\pi\rho_b\int_0^{R_0} UR\,dR - 2\pi\rho_b\beta\int_0^{R_0}(T-T_b)UR\,dR$$

$$= 2\pi\rho_b\int_0^{R_0} UR\,dR = \pi R_0^2\rho_b U_m. \tag{12}$$

Let us define the dimensionless quantities

$$u = \frac{U}{U_m}, \quad \theta = \frac{T-T_b}{\Delta T}, \quad r = \frac{R}{R_0},$$

$$\lambda = -\frac{R_0^2}{\mu U_m}\frac{dP}{dX}, \quad Gr = \frac{8R_0^3 g\beta\Delta T}{\nu^2}, \quad Re = \frac{2R_0 U_m}{\nu},$$

$$\eta = \frac{T_b - T_0}{\Delta T}, \quad \Xi = \frac{Gr}{Re} \tag{13}$$

where the reference temperature difference ΔT is given by

$$\Delta T = \frac{\mu U_m^2}{k}. \tag{14}$$

Although Gr is always positive, Re and Ξ can be either positive or negative. In particular, in the case of upward flow ($U_m > 0$), both Re and Ξ are positive, while, for downward flow ($U_m < 0$), these dimensionless parameters are negative. As a consequence of equations (13) and (14), equations (5) and (9)–(11) can be rewritten as

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = -\frac{\Xi}{4}\theta - \lambda \tag{15}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) = -\left(\frac{du}{dr}\right)^2 \tag{16}$$

$$u(1) = 0, \quad \theta(1) = -\eta \tag{17}$$

$$\left.\frac{du}{dr}\right|_{r=0} = 0, \quad \left.\frac{d\theta}{dr}\right|_{r=0} = 0. \tag{18}$$

Moreover, equations (3), (4) and (13) yield

$$\int_0^1 u\theta r\,dr = 0 \tag{19}$$

$$\int_0^1 ur\,dr = \frac{1}{2}. \tag{20}$$

For any prescribed value of the dimensionless parameter Ξ, equations (15)–(20) allow one to determine the functions $u(r)$ and $\theta(r)$ as well as the constants λ and η. In particular, it is easily verified that the choice $\Xi = 0$ corresponds to the absence of buoyancy forces, i.e. to forced convection.

As a consequence of equations (19) and (20), equation (15) yields

$$\int_0^1 u\frac{d}{dr}\left(r\frac{du}{dr}\right)dr = -\frac{\lambda}{2}. \tag{21}$$

By employing equations (17), (18) and an integration by parts, one can rewrite equation (21) in the form

$$\int_0^1 r\left(\frac{du}{dr}\right)^2 dr = \frac{\lambda}{2}. \tag{22}$$

Then, as a consequence of equations (16) and (18), equation (22) yields

$$\left.\frac{d\theta}{dr}\right|_{r=1} = -\frac{\lambda}{2}. \tag{23}$$

The Nusselt number is defined as

$$Nu \equiv \frac{2R_0}{T_0 - T_b}\left.\frac{dT}{dR}\right|_{R=R_0}. \tag{24}$$

Equations (13), (17) and (23) imply that equation (24) can be rewritten as

$$Nu = \frac{2}{\theta(1)}\left.\frac{d\theta}{dr}\right|_{r=1} = \frac{\lambda}{\eta}. \tag{25}$$

Moreover, the Brinkman number is defined as

$$Br \equiv \frac{\mu U_m^2}{k[T(0) - T_0]} \tag{26}$$

so that equations (13), (14) and (17) yield

$$Br = \frac{1}{\theta(0) + \eta}. \tag{27}$$

If the condition of uniform wall temperature T_0 is accomplished by an external convection with a fluid having a reference temperature T_e and a heat transfer coefficient h_e, then the heat flux per unit area at $R = R_0$ is given by

$$-k\left.\frac{dT}{dR}\right|_{R=R_0} = h_e(T_0 - T_e). \tag{28}$$

As a consequence of equations (13) and (23), equation (28) can be rewritten as

$$\eta_e = \eta + \frac{\lambda}{2Bi} \tag{29}$$

where the Biot number is given by $Bi = h_e R_0/k$, while η_e is a dimensionless parameter such that $\eta_e = (T_b - T_e)/\Delta T$. Obviously, in the limit $Bi \to \infty$, equation (29) yields $\eta = \eta_e$, so that the prescribed reference temperature of the external fluid T_e coincides with the wall temperature. Then, in the limit $Bi \to \infty$, the convective boundary condition collapses to the boundary condition of a prescribed wall temperature. The mixed convection problem with a convective boundary condition can be solved as follows. First, for the given value of Ξ, equations (15)–(20) yield the functions $u(r)$, $\theta(r)$ as well as the constants λ and η. Then, for the given value of Bi, equation (29) yields the value of η_e. As a consequence, $u(r)$, $\theta(r)$, λ and η are independent of Bi. Moreover, on account of equations (25) and (27), also the values of Nu and Br are independent of the Biot number and, hence, they coincide with the values for $Bi \to \infty$, i.e. for a prescribed wall temperature. Therefore, the fully developed Nusselt number is independent of Bi not only in the case of forced convection, as is shown in ref. [3], but also in the case of mixed convection.

3. Perturbation series solution

In this section, the perturbation method which leads to a solution of equations (15)–(20) is described.

Let us expand the functions $u(r)$, $\theta(r)$ and the constants λ and η as power series in the parameter Ξ, namely

$$u(r) = u_0(r) + u_1(r)\Xi + u_2(r)\Xi^2 + \cdots = \sum_{n=0}^{\infty} u_n(r)\Xi^n \tag{30}$$

$$\theta(r) = \theta_0(r) + \theta_1(r)\Xi + \theta_2(r)\Xi^2 + \cdots = \sum_{n=0}^{\infty} \theta_n(r)\Xi^n \tag{31}$$

$$\lambda = \lambda_0 + \lambda_1\Xi + \lambda_2\Xi^2 + \cdots = \sum_{n=0}^{\infty} \lambda_n\Xi^n \tag{32}$$

$$\eta = \eta_0 + \eta_1\Xi + \eta_2\Xi^2 + \cdots = \sum_{n=0}^{\infty} \eta_n\Xi^n. \tag{33}$$

These power-series expansions are substituted in equations (15)–(20). After substitution, one collects the terms having like powers of Ξ and equates to zero the coefficient of each power of Ξ. Then, the original boundary value problem is mapped into a sequence of boundary value problems which can be solved in succession, in order to obtain the coefficients of the power-series expansions reported in equations (30)–(33). A detailed analysis on the application of perturbation methods in heat transfer problems can be found, for instance, in the book by Aziz and Na [16].

The boundary value problem which corresponds to $n = 0$ is the following

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_0}{dr}\right) = -\lambda_0 \tag{34}$$

$$\left.\frac{du_0}{dr}\right|_{r=0} = 0, \quad u_0(1) = 0 \tag{35}$$

$$\int_0^1 u_0 r\, dr = \frac{1}{2} \tag{36}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta_0}{dr}\right) = -\left(\frac{du_0}{dr}\right)^2 \tag{37}$$

$$\left.\frac{d\theta_0}{dr}\right|_{r=0} = 0, \quad \theta_0(1) = -\eta_0 \tag{38}$$

$$\int_0^1 u_0\theta_0 r\, dr = 0. \tag{39}$$

Equations (34)–(39) are easily solved, because the function $\theta_0(r)$ does not affect the function $u_0(r)$. The latter, together with the constant λ_0, is easily determined by solving equations (34)–(36), namely

$$u_0(r) = 2(1-r^2), \quad \lambda_0 = 8. \tag{40}$$

By substituting equation (40) in equations (37)–(39), one obtains the function $\theta_0(r)$ and the constant η_0, which are given by

$$\theta_0(r) = \tfrac{1}{6} - r^4, \quad \eta_0 = \tfrac{5}{6}. \tag{41}$$

Equations (30)–(33) reveal that $\theta_0(r)$ and $u_0(r)$ are the dimensionless temperature and velocity profiles in the case $\Xi = 0$, i.e. in the case of forced convection. As expected, $u_0(r)$ is the usual Hagen–Poiseuille velocity profile of laminar forced convection. Obviously, also λ_0 and η_0 are the values of λ and η in the case $\Xi = 0$. Therefore, in this case, equations (25), (40) and (41) yield $Nu = \lambda_0/\eta_0 = 48/5 = 9.6$, which is the well known value of Nu for the case of laminar forced convection discussed in the Introduction.

The boundary value problem which corresponds to every $n > 0$ is the following

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_n}{\mathrm{d}r}\right) = -\frac{1}{4}\theta_{n-1} - \lambda_n \tag{42}$$

$$\left.\frac{\mathrm{d}u_n}{\mathrm{d}r}\right|_{r=0} = 0, \quad u_n(1) = 0 \tag{43}$$

$$\int_0^1 u_n r\,\mathrm{d}r = 0 \tag{44}$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\theta_n}{\mathrm{d}r}\right) = -\sum_{j=0}^{n}\frac{\mathrm{d}u_j}{\mathrm{d}r}\frac{\mathrm{d}u_{n-j}}{\mathrm{d}r} \tag{45}$$

$$\left.\frac{\mathrm{d}\theta_n}{\mathrm{d}r}\right|_{r=0} = 0, \quad \theta_n(1) = -\eta_n \tag{46}$$

$$\sum_{j=0}^{n}\int_0^1 u_j\theta_{n-j} r\,\mathrm{d}r = 0. \tag{47}$$

If the functions $u_j(r)$ and $\theta_j(r)$ are known for every j such that $0 \leqslant j \leqslant n-1$, then equations (42)–(47) allow one to obtain $u_n(r)$, $\theta_n(r)$ and the constants λ_n and η_n. In particular, as a consequence of equations (42)–(44), $u_n(r)$ and λ_n are given by

$$u_n(r) = \frac{1}{4}\int_r^1 \frac{1}{r''}\left[\int_0^{r''}\theta_{n-1}(r')r'\,\mathrm{d}r'\right]\mathrm{d}r'' + \frac{\lambda_n}{4}(1-r^2) \tag{48}$$

$$\lambda_n = -4\int_0^1 r\left\{\int_r^1 \frac{1}{r''}\left[\int_0^{r''}\theta_{n-1}(r')r'\,\mathrm{d}r'\right]\mathrm{d}r''\right\}\mathrm{d}r. \tag{49}$$

Then, equations (45)–(47) yield $\theta_n(r)$ and η_n, namely

$$\theta_n(r) = -\eta_n + \int_r^1 \frac{1}{r''}\left[\int_0^{r''}\sum_{j=0}^{n}\frac{\mathrm{d}u_j(r')}{\mathrm{d}r'}\frac{\mathrm{d}u_{n-j}(r')}{\mathrm{d}r'}r'\,\mathrm{d}r'\right]\mathrm{d}r'' \tag{50}$$

$$\eta_n = 2\int_0^1 u_0(r)r\left\{\int_r^1 \frac{1}{r''}\left[\int_0^{r''}\sum_{j=0}^{n}\frac{\mathrm{d}u_j(r')}{\mathrm{d}r'}\times\frac{\mathrm{d}u_{n-j}(r')}{\mathrm{d}r'}r'\,\mathrm{d}r'\right]\mathrm{d}r''\right\}\mathrm{d}r + 2\int_0^1\left[\sum_{j=1}^{n}u_j(r)\theta_{n-j}(r)\right]r\,\mathrm{d}r. \tag{51}$$

By employing equations (40), (41) and (48)–(51), one can evaluate the functions $u_n(r)$, $\theta_n(r)$ and the constants λ_n and η_n for any arbitrary n. Although the expansions of $u(r)$, $\theta(r)$, λ and η given by equations (30)–(33) consist of an infinite number of terms, in practice one can only deal with truncated perturbation series. In particular, if only the first three terms of each series are considered, one obtains

$$u(r) \cong 2(1-r^2) + \frac{\Xi}{48}\left(\frac{1}{6} - \frac{r^2}{2} + \frac{r^6}{3}\right) + \frac{\Xi^2}{1536}\left(\frac{19}{450} - \frac{2r^2}{15} + \frac{r^6}{9} - \frac{r^{10}}{50}\right) \tag{52}$$

$$\theta(r) \cong \frac{1}{6} - r^4 + \frac{\Xi}{96}\left(\frac{19}{180} - r^4 + \frac{r^8}{2}\right) + \frac{\Xi^2}{12\,288}\left(\frac{233}{1890} - \frac{7r^4}{5} + r^8 - \frac{32r^{12}}{135}\right) \tag{53}$$

$$\lambda \cong 8 + \frac{\Xi^2}{13\,824} \tag{54}$$

$$\eta \cong \frac{5}{6} + \frac{71\Xi}{17\,280} + \frac{971\Xi^2}{23\,224\,320}. \tag{55}$$

4. Discussion of the results

In this section, 40-terms perturbation series are employed to evaluate the dimensionless velocity profile and the dimensionless temperature profile.

When a perturbation method is employed, it is quite important to determine the domain of the perturbation parameter Ξ where the perturbation expansions are regular. In other words, one should evaluate or at least estimate the radius of convergence of the power series given by equations (30)–(33). A technique to obtain the maximum value of Ξ which yields convergent perturbation series is based on the estimate of D'Alembert's ratio limit by means of the Domb–Sykes plots [16].

In Figs 2 and 3, the sequences $\log_{10}|\lambda_{n-1}/\lambda_n|$, $\log_{10}|\eta_{n-1}/\eta_n|$, $\log_{10}|u_{n-1}(0)/u_n(0)|$ and $\log_{10}|\theta_{n-1}(0)/\theta_n(0)|$ are plotted vs. $1/n$. These Domb–Sykes plots show that all these sequences have the same limiting value for $n \to \infty$ and that this limiting value can be estimated as 1.786. As a consequence, the radius of convergence of the perturbation series is $10^{1.786} \cong 61$. Then, the perturbation expansions given by equations (30)–(33) can be considered as regular in the range $|\Xi| < 61$.

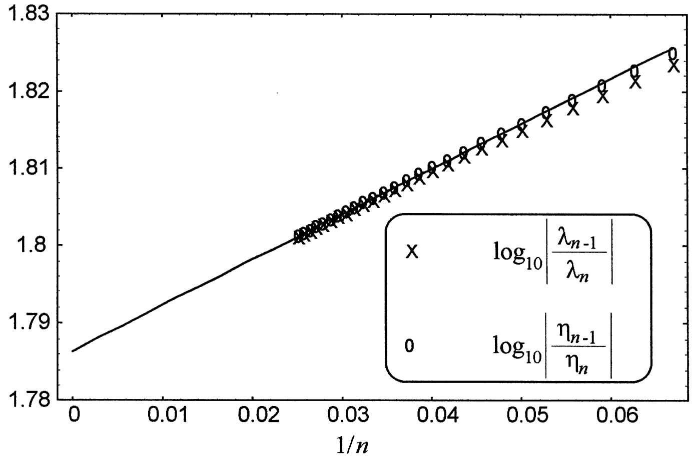

Fig. 2. Domb–Sykes plots for the perturbation expansions given by equations (32) and (33).

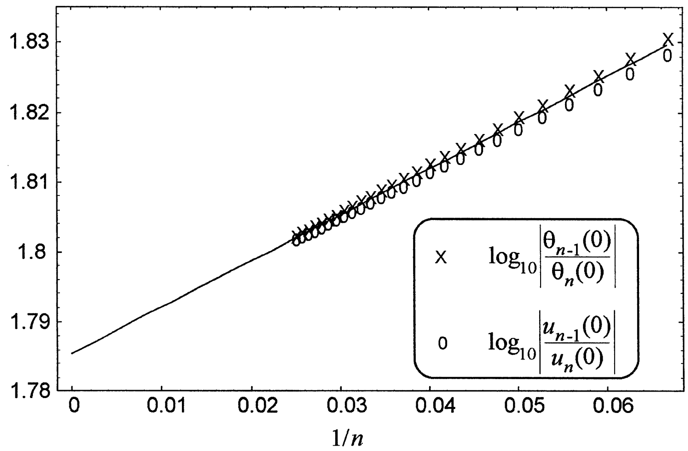

Fig. 3. Domb–Sykes plots for the perturbation expansions given by equations (30) and (31) with $r = 0$.

As a consequence of equation (30), the value of u at $r = 0$ is given by

$$u(0) = \lim_{n\to\infty} S_n(\Xi) \tag{56}$$

where the partial sum $S_n(\Xi)$ is defined as

$$S_n(\Xi) \equiv \sum_{j=0}^{n} u_j(0)\Xi^j. \tag{57}$$

Plots of $S_n(\Xi)$ vs. n are reported in Fig. 4 for $\Xi = 55$ and $\Xi = -55$. These plots reveal that the convergence of the perturbation series is different in the two cases $\Xi > 0$ and $\Xi < 0$. In particular, the sequence of the partial sums $S_n(\Xi)$ is monotonic for $\Xi = 55$, while this sequence is oscillating for $\Xi = -55$.

In Table 1, values of λ_n, η_n and $\theta_n(0)$ are reported for $n \leqslant 30$. These values allow one to evaluate λ, η, Nu and Br, by employing equations (25), (27) and (31)–(33). In Table 2, the values of λ, η, Nu and Br are obtained by means of 40-terms perturbation series, for values of Ξ which lie in the interval $-55 \leqslant \Xi \leqslant 55$. Table 2 reveals that the buoyancy forces affect the dimensionless parameters λ, η, Nu and Br as follows. The pressure drop parameter λ is increased both for downward flow ($\Xi < 0$) and for upward flow ($\Xi > 0$). The temperature difference ratio η is decreased for $\Xi < 0$ and is increased for $\Xi > 0$. On the other hand, both the Nusselt number Nu and the Brinkman number Br are increased for downward flow and are decreased for upward flow.

To summarize, for downward flow, the buoyancy effect tends to increase the convection coefficient and the pressure drop. Moreover, for downward flow, this effect tends to decrease both the bulk temperature and the temperature on the axis of the tube. On the other hand, for upward flow, the bulk temperature, the temperature on the axis and the pressure drop are increased by the buoyancy effect, while the convection coefficient is decreased.

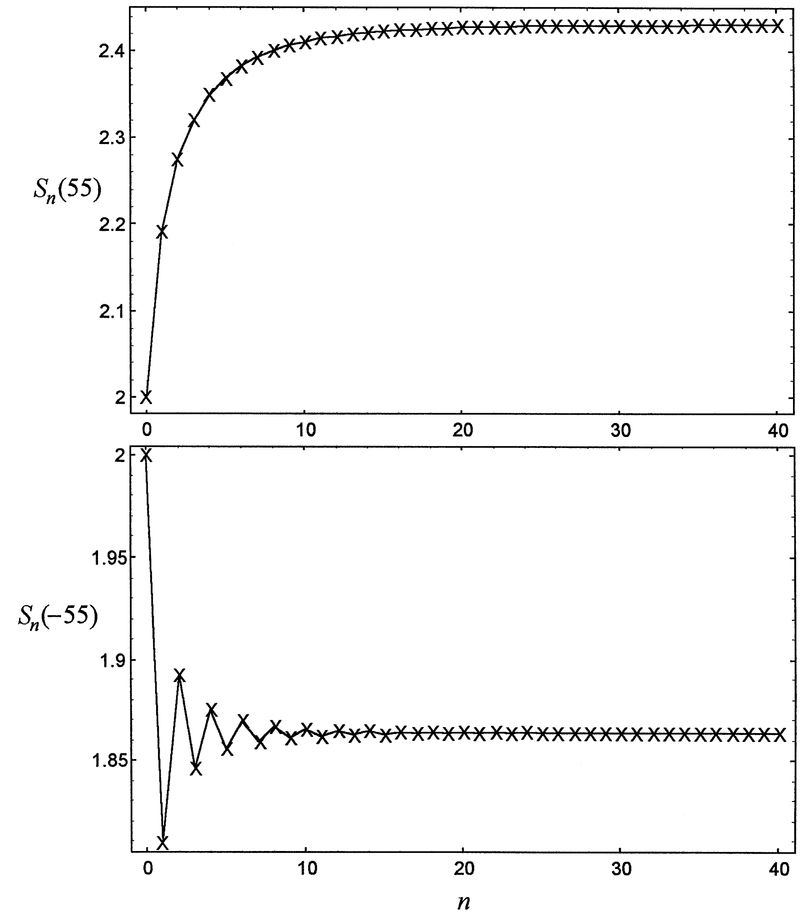

Fig. 4. Plots of $S_n(\Xi)$ vs. n for $\Xi = 55$ and for $\Xi = -55$.

For downward flow, the values of Nu and Br can be approximated by means of the correlations

$$Nu = 9.60336 + 4.62883 \cdot 10^{-2}|\Xi| - 1.02596 \cdot 10^{-4}|\Xi|^2 \tag{58}$$

$$Br = 1.00055 + 5.03746 \cdot 10^{-3}|\Xi| - 1.50116 \cdot 10^{-5}|\Xi|^2. \tag{59}$$

These correlations can be employed in the range $-55 \leqslant \Xi \leqslant 0$ with a relative error lower than 0.04%. On the other hand, for upward flow, the following correlations for Nu and Br can be employed:

$$Nu = 9.61764 - 5.18076 \cdot 10^{-2}\Xi - 8.27195 \cdot 10^{-8}\Xi^4 \tag{60}$$

$$Br = 1.00244 - 5.86703 \cdot 10^{-3}\Xi - 1.3985 \cdot 10^{-8}\Xi^4. \tag{61}$$

In the range $0 \leqslant \Xi \leqslant 55$, the values of Nu and Br given by equations (60) and (61) agree with those obtained by means of 40-terms perturbation series with a relative error lower than 0.31%.

In Figs 5–7, the dimensionless velocity profile and the dimensionless temperature profile are plotted for $\Xi = 55$, $\Xi = 40$ and $\Xi = -55$, respectively. The plots have been obtained by employing 40-terms perturbation series. In each of these figures, a comparison between the behaviour of u and θ in the case of mixed convection and the behaviour of u and θ in the case of forced convection is performed. Figures 5 and 6 reveal that, for upward flow, the buoyancy effect increases the dimensionless velocity and the dimensionless temperature on the axis, while this effect decreases both u and θ in the neighbourhood of the wall. This circumstance is more apparent for $\Xi = 55$ than for $\Xi = 40$. Indeed, the fluid temperature is higher on the axis than on the wall, so that the mass density is smaller on the axis than on the wall. Therefore, for $\Xi > 0$, the flow is assisted by the buoyancy effect in the neighbourhood of the axis, while the flow is inhibited near the wall. Obviously, the reverse occurs for $\Xi < 0$. In fact, Fig. 7 shows that, in the neighbourhood of the axis, both u and θ are smaller for $\Xi = -55$ than for $\Xi = 0$. On the other hand, this figure reveals that, near the wall, both u and θ are increased by the buoyancy effect.

Table 1
Values of λ_n, η_n and $\theta_n(0)$ for $n \leqslant 30$

n	λ_n	η_n	$\theta_n(0)$
0	8	0.8333	0.1666
1	0	4.109×10^{-3}	1.100×10^{-3}
2	7.234×10^{-5}	4.181×10^{-5}	1.003×10^{-5}
3	1.055×10^{-6}	4.836×10^{-7}	1.060×10^{-7}
4	1.439×10^{-8}	5.994×10^{-9}	1.223×10^{-9}
5	1.966×10^{-10}	7.772×10^{-11}	1.498×10^{-11}
6	2.721×10^{-12}	1.041×10^{-12}	1.916×10^{-13}
7	3.819×10^{-14}	1.428×10^{-14}	2.531×10^{-15}
8	5.428×10^{-16}	1.997×10^{-16}	3.428×10^{-17}
9	7.803×10^{-18}	2.835×10^{-18}	4.736×10^{-19}
10	1.133×10^{-19}	4.076×10^{-20}	6.651×10^{-21}
11	1.659×10^{-21}	5.923×10^{-22}	9.469×10^{-23}
12	2.447×10^{-23}	8.684×10^{-24}	1.363×10^{-24}
13	3.635×10^{-25}	1.283×10^{-25}	1.982×10^{-26}
14	5.431×10^{-27}	1.908×10^{-27}	2.907×10^{-28}
15	8.156×10^{-29}	2.855×10^{-29}	4.293×10^{-30}
16	1.231×10^{-30}	4.294×10^{-31}	6.383×10^{-32}
17	1.865×10^{-32}	6.488×10^{-33}	9.544×10^{-34}
18	2.837×10^{-34}	9.845×10^{-35}	1.434×10^{-35}
19	4.330×10^{-36}	1.499×10^{-36}	2.165×10^{-37}
20	6.631×10^{-38}	2.291×10^{-38}	3.282×10^{-39}
21	1.018×10^{-39}	3.512×10^{-40}	4.994×10^{-41}
22	1.568×10^{-41}	5.400×10^{-42}	7.625×10^{-43}
23	2.420×10^{-43}	8.323×10^{-44}	1.168×10^{-44}
24	3.744×10^{-45}	1.286×10^{-45}	1.793×10^{-46}
25	5.805×10^{-47}	1.991×10^{-47}	2.762×10^{-48}
26	9.018×10^{-49}	3.089×10^{-49}	4.263×10^{-50}
27	1.403×10^{-50}	4.803×10^{-51}	6.595×10^{-52}
28	2.188×10^{-52}	7.480×10^{-53}	1.022×10^{-53}
29	3.416×10^{-54}	1.167×10^{-54}	1.588×10^{-55}
30	5.341×10^{-56}	1.823×10^{-56}	2.471×10^{-57}

Table 2
Values of λ, η, Nu and Br for various values of Ξ

Ξ	λ	η	Nu	Br
−55	8.119	0.6856	11.84	1.233
−50	8.103	0.6948	11.66	1.215
−45	8.087	0.7046	11.48	1.197
−40	8.072	0.7151	11.29	1.178
−35	8.058	0.7262	11.10	1.158
−30	8.045	0.7381	10.90	1.138
−25	8.033	0.7510	10.70	1.117
−20	8.022	0.7648	10.49	1.096
−15	8.013	0.7797	10.28	1.073
−10	8.006	0.7960	10.06	1.050
−5	8.002	0.8138	9.833	1.025
0	8.000	0.8333	9.600	1.000
5	8.002	0.8550	9.359	0.9733
10	8.008	0.8792	9.109	0.9452
15	8.021	0.9064	8.849	0.9156
20	8.041	0.9374	8.577	0.8840
25	8.070	0.9732	8.292	0.8503
30	8.113	1.015	7.991	0.8140
35	8.176	1.066	7.670	0.7745
40	8.266	1.129	7.325	0.7308
45	8.401	1.209	6.946	0.6814
50	8.616	1.322	6.519	0.6236
55	9.004	1.498	6.009	0.5509

Regarding Table 2 and Figs 5–7, it can be pointed out that the changes induced by buoyancy on the velocity profile and on the temperature profile are more apparent in the case of upward flow than in the case of downward flow. In any case, these changes are not substantial unless $|\Xi| \gtrsim 5$.

The following remark could be useful. The dimensionless velocity profile $u(r)$, the dimensionless temperature profile $\theta(r)$ and the values of λ, η, Nu and Br are uniquely determined by the dimensionless parameter Ξ. The effect of buoyancy is more and more relevant as $|\Xi|$ increases. Indeed, as a consequence of equations (12)–(14), Ξ can be expressed as $\Xi = M/M_0$, where $M_0 = \pi k/(4g\beta)$ is a reference mass flow rate. Then, for a given value of the mass flow rate M, the value of $|\Xi|$ increases as M_0 decreases. In other words, for a prescribed value of M, the buoyancy effect is more important for fluids with a small value of M_0, i.e. for fluids with a small thermal conductivity k and a high coefficient of thermal expansion β. In fact, if the thermal conductivity is small, the power generated by viscous dissipation within the fluid is hardly released to the external environment. As a consequence, appreciable temperature differences are present within the fluid and, if β is sufficiently high, the buoyancy effect influences both the velocity profile and the temperature profile.

A final remark on the mathematical model employed in this paper is as follows. The bulk temperature T_b has been employed as the reference temperature in the Boussinesq approximation. Then, all the thermophysical properties k, α, μ, β and ρ_b must be evaluated at the temperature T_b, which is not known a priori. As a consequence, also the value of $\Xi = Gr/Re$ cannot be known a priori, so that a trial and error method should be employed. In practice, one could adopt the following procedure: prescribe the expected value of T_b; determine the corresponding values of k, μ, β and ρ_b; evaluate Ξ and ΔT; apply the perturbation method to obtain the temperature difference ratio η; verify if the prescribed value of T_b is equal to $T_0+\eta\Delta T$; stop if the equality is satisfied with an acceptable accuracy, otherwise restart the procedure with a new value of T_b. However, it should be pointed out that the mathematical model is based on the hypothesis that the thermophysical properties ρ_b, k, α, μ and β have a very weak dependence on temperature, so that they can be treated as constants. Then, in order to obtain Ξ and ΔT, one can determine the values of the

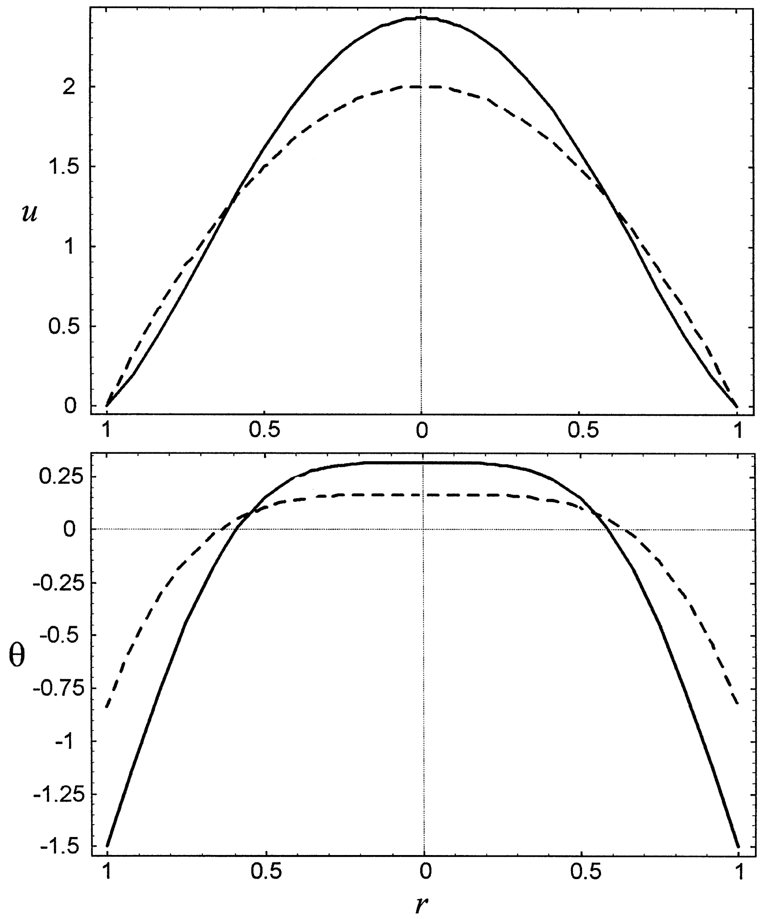

Fig. 5. Plots of $u(r)$ and $\theta(r)$ for $\Xi = 55$ (solid lines) and for $\Xi = 0$ (dashed lines).

properties k, μ, β and ρ_b at the temperature T_0 instead of T_b. In the range of applicability of the mathematical model, the error is expected to be negligible.

5. Conclusions

The stationary and laminar convection in a vertical circular tube with a uniform wall temperature has been studied by taking into account both the viscous dissipation effect and the buoyancy effect. It has been assumed that the velocity field is parallel to the axis of the tube. The mass flow rate has been considered as prescribed and the bulk temperature has been chosen as the reference fluid temperature in the Boussinesq approximation. The momentum balance equation and the energy balance equation have been written in a dimensionless form such that the dimensionless velocity u and the dimensionless temperature θ are uniquely determined by the ratio $\Xi = Gr/Re$.

A perturbation method has been employed to evaluate the dimensionless velocity u, the dimensionless temperature θ, the Nusselt number Nu, the Brinkman number Br, the pressure drop parameter λ and the temperature difference ratio η. By means of the Domb–Sykes plots, it has been shown that the perturbation expansions can be considered as regular in the range $|\Xi| < 61$. Moreover, 40-terms perturbation series have been used to obtain the values of Nu, Br, λ and η as well as to plot the functions $u(r)$ and $\theta(r)$ for some values of Ξ. It has been pointed out that the changes induced by the buoyancy effect on u and θ are more relevant for upward flow than for downward flow. In any case, no substantial difference between the forced convection solution and the mixed convection solution has been found unless $|\Xi| \gtrsim 5$.

Also the case of convective boundary conditions with

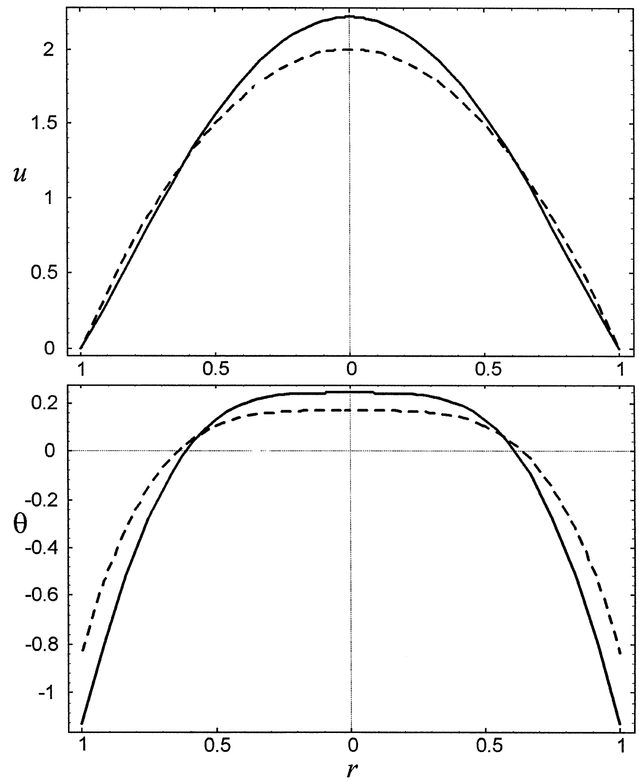

Fig. 6. Plots of $u(r)$ and $\theta(r)$ for $\Xi = 40$ (solid lines) and for $\Xi = 0$ (dashed lines).

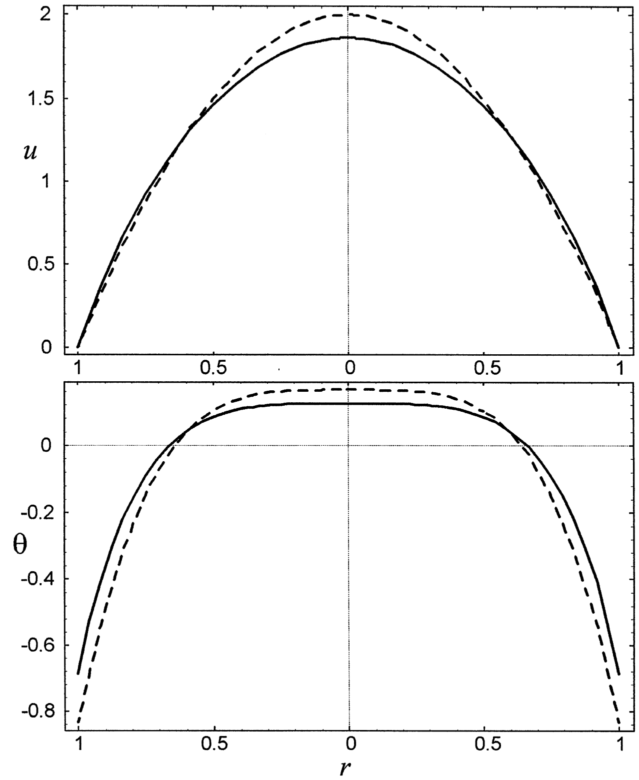

Fig. 7. Plots of $u(r)$ and $\theta(r)$ for $\Xi = -55$ (solid lines) and for $\Xi = 0$ (dashed lines).

a given value of the Biot number *Bi* has been discussed. It has been pointed out that u, θ, *Nu* and *Br* are independent of *Bi*. Hence, the values of u, θ, *Nu* and *Br* for convective boundary conditions coincide with those for a prescribed wall temperature. Indeed, the latter case corresponds to a convective boundary condition with $Bi \to \infty$.

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